

## Introducing small-world network effects to critical dynamics

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We analytically investigate the kinetic Gaussian model and the one-dimensional kinetic Ising model of two typical small-world networks (SWN), the adding type and the rewiring type. The general approaches and some basic equations are systematically formulated. The rigorous investigation of the Glauber-type kinetic Gaussian model shows the mean-field-like global influence on the dynamic evolution of the individual spins. Accordingly a simplified method is presented and tested, which is believed to be a good choice for the mean-field transition widely (in fact, without exception so far) observed for SWN. It yields the evolving equation of the Kawasaki-type Gaussian model. In the one-dimensional Ising model, the  $p$  dependence of the critical point is analytically obtained and the nonexistence of such a threshold  $p_c$ , for a finite-temperature transition, is confirmed. The static critical exponents  $\gamma$  and  $\beta$  are in accordance with the results of the recent Monte Carlo simulations, and also with the mean-field critical behavior of the system. We also prove that the SWN effect does not change the dynamic critical exponent  $z=2$  for this model. The observed influence of the long-range randomness on the critical point indicates two obviously different hidden mechanisms.

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### I. INTRODUCTION

Small-world networks (SWNs) are networks intermediate between a regular lattice and a random graph (see Refs. [1–4] and references therein for review). They are believed to capture the essence of networks in reality, such as neural networks [5], power grids, social networks, and documents on the World Wide Web [6,7], where remote vertices, while locally clustered, often have the chance to be connected via shortcuts. Since 1998, when Watts and Strogatz presented a simple model showing SWN effects [1], it has been studied intensively and extensively [8]. From the point of view of statistical physics, the presence of shortcuts assists the system to behave as a whole, showing global coherence and SWN behavior.

One may be curious about the extent to which the features of phase transitions will be different in spin-lattice models built on small-world networks. Because the SWN effect widely exists in reality, this question is also of much significance. Although new and interesting features have been revealed recently [4,9–16], it is still far from being completely answered. The dynamic aspect has been even less well understood [17]. Naturally, we may expect the evolution of a single spin to be influenced partly by the overall system, however, we find it difficult to offer more specific information. This is due to the complexity of the dynamics itself, and the often formidable mathematical task.

In this paper, we report our work on this interesting problem—the critical dynamics of spin-lattice models built on SWN. As the introductory content, we shall first discuss the general approach. Then, among the various model systems, we choose two special ones for a detailed investigation: the Gaussian model, which is relatively easy in math-

ematics, and the one-dimensional Ising model. Interesting kinetic features are revealed analytically and the study in the dynamic aspect also yields much information about the static properties, such as the critical point.

This paper is organized as follows: In Sec. II we give the definition of spin-lattice models built on SWN. Section III contains the discussion of the general approach (along with a brief review of the dynamic mechanisms). The direct application to the kinetic Gaussian model can be found in Sec. IV. In Sec. V, we present a simplified method and prove its validity in Sec. V A by comparing its result with the rigorous one obtained in Sec. IV. Sections V B and V C are devoted to the further applications of this method on the Kawasaki-type Gaussian model and the one-dimensional Glauber-type Ising model, respectively. In Sec. VI, the influence of the randomness on the critical point is analyzed. Section VII is the summarization with some discussions.

### II. SPIN-LATTICE MODELS BUILT ON SMALL-WORLD NETWORKS

Following the first prototype [1] of SWN, there have been a number of variants (see, for example, Ref. [18]) in two basic groups, which can be constructed as the following: the initial network is, for example, a one-dimensional loop of  $N$  vertices, each vertex being connected to its  $2k$  nearest neighbors. (1) Each pair of random vertices is additionally connected with probability  $p_A$ ; (2) the vertices are then visited one after the other, and each link connecting a vertex to one of its  $k$  nearest neighbors in the clockwise sense is left in place with probability  $1-p_R$ , and with probability  $p_R$  it is reconnected to another randomly chosen vertex. We may call the first group adding-type (SWNs) and the second group rewiring-type (SWNs). Both of these modifications introduce long-range connections. The above algorithms can be extended to systems with higher dimensionality and even frac-

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tal structures. (For example, for a spin located somewhere in a cubic lattice, three of the six bonds in some fixed directions will be reconnected with probability  $p_R$ , respectively.) These constructions, with any (infinitesimally low) fraction of shortcuts, allow us to reconcile local properties of a regular network (clustering effect) with global properties of a random one (the average distance  $l \sim \log_{10} N$ ).

A spin-lattice model built on SWN, which is the focus of this paper, can thus be defined: In a  $D$ -dimensional regular network consisting of  $N$  spins with periodic boundary condition, each spin is linked to its  $2kD$  nearest neighbors (in this paper we will choose  $k=1$ ). Then (1) a certain number of supplemental links are added; (2) a portion of the bonds are rewired. Whether it is in group (1) (adding-type SWN) or group (2) (rewiring-type SWN), two spins connected by a shortcut act in the same way as those connected by a regular bond, that is, their interaction contributes to the system Hamiltonian and they obey the redistribution (exchange) mechanism. In short, there is no difference between a long-range bond and a regular one. Other definitions may exist. For example, since these small-world networks are model systems for the networks in reality, one may have a reason to choose certain physical quantities, such as  $J$  in the Ising model, to be different. However, although it is not necessary, we will limit the scope of this paper to the above rule.

### III. THE DYNAMIC MECHANISM

The various dynamic processes in the critical phenomena are believed to be governed by two basic mechanisms, Glauber type with order parameter nonconserved and Kawasaki type with order parameter conserved. Recently, a Glauber-type single-spin transition mechanism [19,20] and a Kawasaki-type spin-pair redistribution mechanism [21,22] have been presented as the natural generalizations of Glauber's flipping mechanism [23] and Kawasaki's exchange mechanism [24], respectively. They are generally applicable and mathematically well organized. Our work begins with a review of these two mechanisms. This is for the better understanding of the calculations in Secs. IV and V, and it also might be a convenient reference for further studies.

#### A. Single-spin transition mechanism

Glauber's single-spin flipping mechanism allows an Ising system to evolve with its spins flipping to their opposite. In the single-spin transition mechanism [19], a single spin  $\sigma_i$  may change itself to any possible values,  $\hat{\sigma}_i$ , and the master equation is

$$\begin{aligned} \frac{d}{dt} P(\{\sigma\}, t) = & - \sum_i \sum_{\hat{\sigma}_i} [W_i(\sigma_i \rightarrow \hat{\sigma}_i) P(\{\sigma\}, t) \\ & - W_i(\hat{\sigma}_i \rightarrow \sigma_i) P(\{\sigma_{j \neq i}, \hat{\sigma}_i, t)]. \end{aligned} \quad (1)$$

The transition probability is in a normalized form determined by a heat Boltzmann factor,

$$\begin{aligned} W_i(\sigma_i \rightarrow \hat{\sigma}_i) = & \frac{1}{Q_i} \exp[-\beta \mathcal{H}(\{\sigma_{j \neq i}, \hat{\sigma}_i)], \\ Q_i = & \sum_{\hat{\sigma}_i} \exp[-\beta \mathcal{H}(\{\sigma_{j \neq i}, \hat{\sigma}_i)], \end{aligned} \quad (2)$$

where  $\beta = 1/k_B T$ . Based on the master equation, Eq. (1), one can prove that,

$$\frac{dq_k(t)}{dt} = -q_k(t) + \sum_{\{\sigma\}} \left[ \sum_{\hat{\sigma}_k} \hat{\sigma}_k W_k(\sigma_k \rightarrow \hat{\sigma}_k) \right] P(\{\sigma\}; t), \quad (3)$$

where  $q_k(t) \equiv \sum \{\sigma\} \sigma_k P(\{\sigma\}; t)$ . There are also corresponding equations for the correlation functions.

#### B. Spin-pair redistribution mechanism

Kawasaki's spin-pair exchange mechanism allows an Ising system to evolve with its nearest neighbors exchanging their spin values. In the spin-pair redistribution mechanism [21], two connected spins  $\sigma_j$  and  $\sigma_l$  may change to any possible values,  $\hat{\sigma}_j$  and  $\hat{\sigma}_l$ , as long as their sum are conserved. The master equation is

$$\begin{aligned} \frac{d}{dt} P(\{\sigma\}, t) = & \sum_{\langle jl \rangle} \sum_{\hat{\sigma}_j, \hat{\sigma}_l} [-W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) P(\{\sigma\}; t) \\ & + W_{jl}(\hat{\sigma}_j \hat{\sigma}_l \rightarrow \sigma_j \sigma_l) \\ & \times P(\{\sigma_{i \neq j}, \sigma_{l \neq k}, \hat{\sigma}_j, \hat{\sigma}_l; t)]. \end{aligned} \quad (4)$$

The redistribution probability is also in a normalized form determined by a heat Boltzmann factor,

$$\begin{aligned} W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) = & \frac{1}{Q_{jl}} \delta_{\sigma_j + \sigma_l, \hat{\sigma}_j + \hat{\sigma}_l} \\ & \times \exp[-\beta \mathcal{H}(\{\sigma_m\}_{m \neq j, l}, \hat{\sigma}_j, \hat{\sigma}_l)], \end{aligned} \quad (5)$$

where the normalization factor  $Q_{jl}$  is

$$Q_{jl} = \sum_{\hat{\sigma}_j, \hat{\sigma}_l} \delta_{\sigma_j + \sigma_l, \hat{\sigma}_j + \hat{\sigma}_l} \exp[-\beta \mathcal{H}(\{\sigma_m\}_{m \neq j, l}, \hat{\sigma}_j, \hat{\sigma}_l)].$$

Based on the master equation, Eq. (4), one can prove that

$$\begin{aligned} \frac{dq_k(t)}{dt} = & -2Dq_k(t) + \sum_{\{\sigma\}} \sum_w \left[ \sum_{\hat{\sigma}_k, \hat{\sigma}_{k+w}} \hat{\sigma}_k \right. \\ & \left. \times W_{k, k+w}(\sigma_k \sigma_{k+w} \rightarrow \hat{\sigma}_k \hat{\sigma}_{k+w}) \right] P(\{\sigma\}; t), \end{aligned} \quad (6)$$

where  $D$  is the dimensionality and  $\sum_w$  stands for the summation taken over the nearest neighbors.

### C. How to apply them to SWN

In the construction of SWN, according to a certain probability, we will have a whole set of possible realizations. So the theoretically correct way of treating dynamic systems built on SWN actually consists of three steps: First, we have to make a full list of all the possible realizations and point out the probability of each one of them. Second, we treat each system respectively (apply the dynamic mechanism, and obtain the master equation and the physical quantities of interest). Third, we give the expectation value with all these results. This is cumbersome, but conceptually straightforward. In Sec. V, we shall discuss whether there is any simplified method. (In fact, there is.)

### IV. KINETIC GAUSSIAN MODEL GOVERNED BY THE GLAUBER-TYPE MECHANISM

We have discussed the general approach right above in Sec. III C, and in this section we will directly perform the calculations according to the three steps, to study the three-dimensional (3D) kinetic Gaussian model governed by the Glauber-type single-spin transition mechanism (see Sec. III A). The calculation to be carried out is very long, but fortunately we can borrow some results from our earlier studies [19,20].

The Gaussian model, proposed by Berlin and Kac, is a continuous-spin model. Its Hamiltonian,

$$-\beta\mathcal{H}=K\sum_{\langle i,j \rangle} \sigma_i\sigma_j, \quad (7)$$

where  $K=J/k_B T$ . The spins  $\sigma_k$  can take any real value between  $(-\infty, +\infty)$ , and the probability of finding a given spin between  $\sigma_k$  and  $\sigma_k+d\sigma_k$  is assumed to be the Gaussian-type distribution,  $f(\sigma_k)d\sigma_k \sim \exp[-(b/2)\sigma_k^2]d\sigma_k$ , where  $b$  is a distribution constant independent of temperature. Thus, the summation for the spin value turns into the integration  $\sum_{\sigma} \rightarrow \int_{-\infty}^{\infty} f(\sigma)d\sigma$ . This model has been studied often as a starting point for investigations of other systems.

Governed by the Glauber-type single-spin transition mechanism, the expectation value of single spin [19] obeys

$$\frac{dq_k(t)}{dt} = -q_k(t) + \frac{K}{b} \sum_{\langle q_k', q_k \rangle} q_{k'}(t). \quad (8)$$

Since in SWN, a spin may have a neighbor located very far from it, in the following calculations the summation is to include every spin that is connected with  $\sigma_k$ . Taking average of Eq. (8), we can obtain the evolution of the magnetization  $M(t) = \sum_k q_k(t)/N$ .

#### A. Kinetic Gaussian model on an adding-type small-world network

First, we treat the 3D kinetic Glauber-type Gaussian model on an adding-type SWN consisting of  $N$  (a very large number) spins. The Kawasaki type is still tractable, but the mathematical task will be more complex. We will leave that till Sec. V. Besides the regular bonds, each pair of spins, no

matter how far apart, is connected via an additional bond with probability  $p_A$ . Actually there are  $2^{N(N-1)/2}$  different networks, each with a given probability. As a model system for the networks in reality, we expect the number of these bonds,  $n \sim N(N-1)p_A/2$ , to be much smaller than  $N$ . So practically we require  $p_A N \ll 1$ . With respect to a specific spin  $\sigma_{ijk}$ , all the networks can be divided into  $N$  groups listed as the following.

(a) (0): There is no random bond on  $\sigma_{ijk}$ , and the probability is  $(1-p_A)^{N-1}$ . According to Eq. (8),

$$\begin{aligned} \frac{d}{dt}q_{ijk}(t) = & -q_{ijk}(t) + \frac{K}{b} \sum_w [q_{i+w,j,k}(t) + q_{i,j+w,k}(t) \\ & + q_{i,j,k+w}(t)]. \end{aligned}$$

(b) (1): There is only one random bond on  $\sigma_{ijk}$ , and the probability is  $C_{N-1}^1 p_A (1-p_A)^{N-2}$ . In fact, this group can be further divided into  $N-1$  subgroups, each corresponding to a specific spin connected to  $\sigma_{ijk}$ , and each with the same probability  $p_A(1-p_A)^{N-2}$ . Averaging them, we get

$$\begin{aligned} \frac{dq_{ijk}(t)}{dt} = & -q_{ijk}(t) + \frac{K}{b} \sum_w [q_{i+w,j,k}(t) + q_{i,j+w,k}(t) \\ & + q_{i,j,k+w}(t)] + \frac{K}{b} M(t). \end{aligned}$$

(c) ( $n$ ): There are  $n$  random bonds on  $\sigma_{ijk}$ , and the probability is  $C_{N-1}^n p_A^n (1-p_A)^{N-n-1}$ . Similarly, in average we get,

$$\begin{aligned} \frac{dq_{ijk}(t)}{dt} = & -q_{ijk}(t) + \frac{K}{b} \sum_w [q_{i+w,j,k}(t) + q_{i,j+w,k}(t) \\ & + q_{i,j,k+w}(t)] + \frac{nK}{b} M(t). \end{aligned}$$

(d) ( $N-1$ ): There are  $N-1$  random bonds on  $\sigma_{ijk}$ , and the probability is  $p_A^{N-1}$ .

$$\begin{aligned} \frac{dq_{ijk}(t)}{dt} = & -q_{ijk}(t) + \frac{K}{b} \sum_w [q_{i+w,j,k}(t) + q_{i,j+w,k}(t) \\ & + q_{i,j,k+w}(t)] + \frac{K}{b} (N-1)M(t). \end{aligned}$$

Thus, over all the realizations,

$$\begin{aligned} \frac{dq_{ijk}(t)}{dt} = & -q_{ijk}(t) + \frac{K}{b} \sum_w [q_{i+w,j,k}(t) + q_{i,j+w,k}(t) \\ & + q_{i,j,k+w}(t)] + \frac{K}{b} \sum_{n=0}^{N-1} C_{N-1}^n p_A^n (1 \\ & - p_A)^{N-n-1} nM. \end{aligned} \quad (9)$$

Fortunately, we find the following relationship:

$$\sum_{n=0}^{N-1} C_{N-1}^n p_A^n (1-p_A)^{N-n-1} n = (N-1)p_A, \quad (10)$$

and thus

$$\begin{aligned} \frac{dq_{ijk}(t)}{dt} = & -q_{ijk}(t) + \frac{K}{b} \sum_w [q_{i+w,j,k}(t) + q_{i,j+w,k}(t) \\ & + q_{i,j,k+w}(t)] + \frac{K}{b} (N-1)p_A M(t), \end{aligned} \quad (11)$$

Similar results can be found in one- and two-dimensional models, and taking average we obtain

$$\frac{dM(t)}{dt} = - \left[ 1 - \frac{2DK}{b} - \frac{K}{b} (N-1)p_A \right] M(t), \quad (12)$$

where  $D$  is the dimensionality. The solution of Eq. (12) is

$$M(t) = M(0) \exp(-t/\tau),$$

where the relaxation time

$$\tau = \frac{1}{1 - K/K_c^A},$$

and the critical point

$$K_c^A = \frac{b}{2D + (N-1)p_A}. \quad (13)$$

It is well known that for the critical point of the regular model,  $K_c^{reg} = b/2D$ , we can clearly see that *the critical temperature will get higher as more long range bonds are added*. Actually, for a vertex located on an adding-type SWN, the number of the long-range bonds is  $n_A \sim (N-1)p_A$ . In the small-world region, where the expected  $n_A$  for most of the vertices is very small, the change of the critical point will be almost unperceivable. (The analysis of this result can be found in Sec. VI.)

### B. Kinetic Gaussian model on a rewiring-type small-world network

Second, we treat the kinetic Gaussian model on a rewiring-type SWN with a characteristic probability  $p_R$ . Because of the length, here we only give the details of the 1D case. With regards to a specific spin  $\sigma_k$ , all the networks can be divided into four major groups listed as the following, (1) both of the two regular bonds on  $\sigma_k$ , connecting  $\sigma_{k-1}$  and  $\sigma_{k+1}$ , are not rewired; (2) the bond connecting  $\sigma_{k-1}$  and  $\sigma_k$  is left unchanged but that connecting  $\sigma_k$  and  $\sigma_{k+1}$  is rewired; (3) the bond connecting  $\sigma_k$  and  $\sigma_{k+1}$  is left unchanged but that connecting  $\sigma_{k-1}$  and  $\sigma_k$  is rewired; (4) both of them are rewired. The final result comes from the summation of all the four parts (see Appendix A for the details):

$$\begin{aligned} \frac{dq_k(t)}{dt} = & -q_k(t) + p_R \frac{2K}{b} M(t) \\ & + (1-p_R) \frac{K}{b} [q_{k-1}(t) + q_{k+1}(t)], \end{aligned} \quad (14)$$

In two- and three-dimensional models, there are  $2^4$  and  $2^6$  major groups, respectively, and they can be treated in the same way. Similar results are obtained:

$$\frac{dM(t)}{dt} = - \left( 1 - \frac{2DK}{b} \right) M(t). \quad (15)$$

Obviously, the relaxation time

$$\tau = \frac{1}{1 - K/K_c^R},$$

where the critical point

$$K_c^R = K_c^{reg} = b/2D. \quad (16)$$

To summarize, we strictly obtain the evolution of the Gaussian model built on SWN. On the adding-type SWN, the critical temperature will get higher as more long range bonds are added, while on the rewiring-type SWN the critical temperature is unchanged. With the dynamic scaling hypothesis  $\tau \sim \xi^z \sim |T - T_c|^{-z\nu}$ , we have  $z\nu = 1$ , where  $z$  is the dynamic critical exponent and  $\nu$  is the correlation length critical exponent. We shall leave the summarization of its properties to Sec. VII. The influence of the long-range bonds on the critical point, which is a rather interesting topic, will be discussed in Sec. VI.

The Gaussian model, being an idealization, often has the value of serving as a starting point for the more general studies. The evolving equations of the individual spins show distinctly the influence of the global coherence, which *automatically takes a mean-field-like form as if it comes from an averaged spin*. This character, as well as some already-proved facts in earlier studies about the static behavior, provides us with some helpful hints for the simplification of the method, which is the topic of the following section.

## V. THE SIMPLIFIED METHOD

Presently, for the critical dynamics on SWN, we still lack a well-established approach that should be both theoretically reliable and practically feasible. The above-mentioned method (and the results) is theoretically rigorous, but it is too complex for the other models, even for the simplest one-dimensional Ising model. In the preceding section, we have pointed out that, in *the dynamic evolution* of the Gaussian model built on SWN, the influence of the system as a whole on individual spins is mean-field-like. On the other hand, recent studies on the Ising model and the X-Y model built on SWN have also shown that the phase transition is of *the mean-field* type (see Refs. [4,9–12,15] and Sec. V C for details). The various model systems built on SWN probably belong to the same mean-field universality class, and they



might be treated with the same dynamic approach. In this problem, which is found to be of the mean-field nature, the following method may be the best choice. We deem all the possible networks as a single one. The effective Hamiltonian of a spin-lattice model built on such a network is defined as the expectation value over all possible realizations. Its effective behavior, e.g., the redistribution between two vertices connected with each other, is also the averaged result. For example, two sites,  $\sigma_i$  and  $\sigma_j$ , are connected with probability  $p$ , then redistribution occurs between them with probability  $pW_{ij}$ . (It is the basic assumption when we are treating a system governed by the Kawasaki-type mechanism.) Then we can directly apply the dynamic mechanisms to this system. This method, which is of the mean-field nature, is certainly more tractable in mathematics. We apply it to the kinetic Gaussian model again and see if it will lead to the same result.

### A. Application in kinetic Gaussian model governed by the Glauber-type mechanism

In the 3D Gaussian model ( $N$  spins in total) built on adding-type SWN with periodic boundary condition, the effective Hamiltonian is

$$-\beta\mathcal{H} = K \sum_{i,j,k} \sigma_{ijk} (\sigma_{i+1,j,k} + \sigma_{i,j+1,k} + \sigma_{i,j,k+1}) + \frac{1}{2} K p_A \sum_{i,j,k} \sigma_{ijk} \sum_{i',j',k'} \sigma_{i'j'k'} - \frac{1}{2} K p_A \sum_{i,j,k} \sigma_{ijk}^2. \quad (17)$$

With the Glauber-type transition mechanism, we begin to derive the single-spin evolving equation according to Eq. (3), along the same line of the calculations as in Ref. [19]. We obtain

$$\sum_{\sigma_{ijk}} \hat{\sigma}_{ijk} W_{ijk}(\sigma_{ijk} \rightarrow \hat{\sigma}_{ijk}) = \frac{K}{b} \left[ \sum_w (\sigma_{i+w,j,k} + \sigma_{ij+w,k} + \sigma_{ij,k+w}) + (N-1)p_A \bar{\sigma} \right], \quad (18)$$

where

$$\bar{\sigma} = \frac{1}{N-1} \left( \sum_{lmn} \sigma_{lmn} - \sigma_{ijk} \right) \approx \frac{1}{N} \sum_{lmn} \sigma_{lmn}.$$

Substituting Eq. (18) into Eq. (3), we get

$$\frac{dq_{ijk}(t)}{dt} = -q_{ijk}(t) + \frac{K}{b} \sum_{w=\pm 1} [q_{i+w,j,k}(t) + q_{ij+w,k}(t) + q_{ij,k+w}(t)] + \frac{K}{b} (N-1)p_A M(t). \quad (19)$$

Similar results can be obtained for one- and two-dimensional models. These results, along with those for  $M(t)$ , are exactly in accordance with what we have obtained in Sec. III B, Eqs. (11) and (12).

In the 3D kinetic Gaussian model built on the rewiring-type SWN, the effective Hamiltonian is,

$$-\beta\mathcal{H} = K(1-p_R) \sum_{i,j,k} \sigma_{ijk} (\sigma_{i+1,j,k} + \sigma_{i,j+1,k} + \sigma_{i,j,k+1}) + K p_R \frac{3}{N} \sum_{i,j,k} \sigma_{ijk} \sum_{i',j',k'} \sigma_{i'j'k'} - K p_R \frac{3}{N} \sum_{i,j,k} \sigma_{ijk}^2. \quad (20)$$

Similar calculations yield

$$\frac{dq_{ijk}}{dt} = -q_{ijk} + p_R \frac{6K}{b} M + (1-p_R) \frac{K}{b} \sum_w (q_{i+w,j,k} + q_{i,j+w,k} + q_{i,j,k+w}).$$

This is also in accordance with the rigorous result, Eqs. (14) and (15). Thus, in the kinetic Gaussian model, this simplified method yields the same results as the rigorous ones obtained with the more complex standard method. As various spin-lattice models, such as the Ising model, are believed to show mean-field behavior on SWN, we believe that this simplified method is able to provide at least qualitatively correct information. In this sense, it is very different from the mean-field approximations taken in other universality classes. In later studies, we will use it to study some more complex problems; the following are two examples.

### B. Application in kinetic Gaussian model governed by a Kawasaki-type mechanism

Now we apply this simplified method to study the diffusion process in the kinetic Gaussian model built on SWN. Although we still have to deal with many complex equations, it is relatively easy compared with the formidable task of the standard approach. In such processes, the system is governed by the Kawasaki-type redistribution mechanism. As already mentioned at the beginning of this section, the system behavior, just as the Hamiltonian, is also averaged over all possible realizations. The basic equations for the Kawasaki-type dynamics listed below are generally applicable in various order-parameter-conserved processes.

(1) On  $D$ -dimensional adding-type SWN: Accordingly, the master equation should be modified as,

$$\begin{aligned} \frac{d}{dt} P(\{\sigma\}, t) &= \sum_{(jl)} \sum_{\hat{\sigma}_j, \hat{\sigma}_l} [-W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) P(\{\sigma\}; t) \\ &+ W_{jl}(\hat{\sigma}_j \hat{\sigma}_l \rightarrow \sigma_j \sigma_l) P(\{\sigma_{i \neq j}, \sigma_{l \neq k}\}, \hat{\sigma}_j, \hat{\sigma}_l; t)] \\ &+ \frac{1}{2} p_A \sum_j \sum_{l \neq j} \sum_{\hat{\sigma}_j, \hat{\sigma}_l} [-W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) \\ &\times P(\{\sigma\}; t) + W_{jl}(\hat{\sigma}_j \hat{\sigma}_l \rightarrow \sigma_j \sigma_l) \\ &\times P(\{\sigma_{i \neq j}, \sigma_{l \neq k}\}, \hat{\sigma}_j, \hat{\sigma}_l; t)], \end{aligned} \quad (21)$$

where the redistribution probability  $W_{jl}(\sigma_j\sigma_l \rightarrow \hat{\sigma}_j\hat{\sigma}_l)$  is of the same form as Eq. (5). With Eq. (21), we can get that

$$\begin{aligned} \frac{dq_k(t)}{dt} &= -2Dq_k(t) + \sum_{\{\sigma\}} \sum_w \left[ \sum_{\hat{\sigma}_k, \hat{\sigma}_{k+w}} \hat{\sigma}_k \right. \\ &\quad \times W_{k,k+w}(\sigma_k\sigma_{k+w} \rightarrow \hat{\sigma}_k\hat{\sigma}_{k+w}) \left. \right] P(\{\sigma\};t) \\ &+ p_A \left\{ -(N-1)q_k(t) + \sum_{\{\sigma\}} \left[ \sum_{l \neq k} \sum_{\hat{\sigma}_k, \hat{\sigma}_l} \hat{\sigma}_k \right. \right. \\ &\quad \times W_{kl}(\sigma_k\sigma_l \rightarrow \hat{\sigma}_k\hat{\sigma}_l) \left. \left. \right] P(\{\sigma\};t) \right\} \\ &\equiv A_k^{(1)} + p_A A_k^{(2)}. \end{aligned} \quad (22)$$

(2) On  $D$ -dimensional rewiring-type SWN: Accordingly, the master equation should be modified as

$$\begin{aligned} \frac{d}{dt} P(\{\sigma\}, t) &= (1-p_R) \sum_{\langle jl \rangle} \sum_{\hat{\sigma}_j, \hat{\sigma}_l} [-W_{jl}(\sigma_j\sigma_l \rightarrow \hat{\sigma}_j\hat{\sigma}_l) \\ &\quad \times P(\{\sigma\};t) + W_{jl}(\hat{\sigma}_j\hat{\sigma}_l \rightarrow \sigma_j\sigma_l) \\ &\quad \times P(\{\sigma_{i \neq j}, \sigma_{l \neq k}\}, \hat{\sigma}_j, \hat{\sigma}_l; t)] + Dp_R \frac{1}{N-1} \\ &\quad \times \sum_j \sum_{l \neq j} \sum_{\hat{\sigma}_j, \hat{\sigma}_l} [-W_{jl}(\sigma_j\sigma_l \rightarrow \hat{\sigma}_j\hat{\sigma}_l) \end{aligned}$$

$$\begin{aligned} &P(\{\sigma\};t) + W_{jl}(\hat{\sigma}_j\hat{\sigma}_l \rightarrow \sigma_j\sigma_l) \\ &\quad \times P(\{\sigma_{i \neq j}, \sigma_{l \neq k}\}, \hat{\sigma}_j, \hat{\sigma}_l; t)]. \end{aligned} \quad (23)$$

The redistribution probability  $W_{jl}(\sigma_j\sigma_l \rightarrow \hat{\sigma}_j\hat{\sigma}_l)$  is of the same form as Eq. (5), and

$$\begin{aligned} \frac{dq_k(t)}{dt} &= (1-p_R) \left\{ -2Dq_k(t) + \sum_{\{\sigma\}} \sum_w \left[ \sum_{\hat{\sigma}_k, \hat{\sigma}_{k+w}} \hat{\sigma}_k \right. \right. \\ &\quad \times W_{k,k+w}(\sigma_k\sigma_{k+w} \rightarrow \hat{\sigma}_k\hat{\sigma}_{k+w}) \left. \left. \right] P(\{\sigma\};t) \right\} \\ &+ \frac{Dp_R}{N-1} \left\{ -(N-1)q_k(t) + \sum_{\{\sigma\}} \left[ \sum_{l \neq k} \sum_{\hat{\sigma}_k, \hat{\sigma}_l} \hat{\sigma}_k \right. \right. \\ &\quad \times W_{kl}(\sigma_k\sigma_l \rightarrow \hat{\sigma}_k\hat{\sigma}_l) \left. \left. \right] P(\{\sigma\};t) \right\} \\ &\equiv (1-p_R)R_k^{(1)} + \frac{Dp_R}{N-1}R_k^{(2)}. \end{aligned} \quad (24)$$

Although  $A_k^{(1,2)}$  and  $R_k^{(1,2)}$  are of the same form, respectively, we use different symbols because they are actually different (determined by the Hamiltonian-dependent redistribution probability  $W$ ).

First we treat the Gaussian model built on the adding-type SWN. The first part,  $A_{ijk}^{(1)}$ , comes from the redistribution between the nearest neighbors. In the 3D case, with Eq. (5) we can obtain

$$\begin{aligned} A_{ijk}^{(1)} &= \frac{1}{2(b+K)} b \{ [(q_{i+1,j,k} - q_{ijk}) - (q_{ijk} - q_{i-1,j,k})] + [(q_{i,j+1,k} - q_{ijk}) - (q_{ijk} - q_{i,j-1,k})] + [(q_{i,j,k+1} - q_{ijk}) \\ &\quad - (q_{ijk} - q_{i,j,k-1})] \} + \frac{K}{2(b+K)} [2(2q_{i-1,j,k} - q_{i-1,j+1,k} - q_{i-1,j-1,k}) + (2q_{i-1,j,k} - q_{ijk} - q_{i-2,j,k}) \\ &\quad + 2(2q_{i+1,j,k} - q_{i+1,j+1,k} - q_{i+1,j-1,k}) + (2q_{i+1,j,k} - q_{ijk} - q_{i+2,j,k}) + 2(2q_{i,j-1,k} - q_{i,j-1,k+1} - q_{i,j-1,k-1}) \\ &\quad + (2q_{i,j-1,k} - q_{ijk} - q_{i,j-2,k}) + 2(2q_{i,j+1,k} - q_{i,j+1,k+1} - q_{i,j+1,k-1}) + (2q_{i,j+1,k} - q_{ijk} - q_{i,j+2,k}) \\ &\quad + 2(2q_{i,j,k-1} - q_{i+1,j,k-1} - q_{i-1,j,k-1}) + (2q_{i,j,k-1} - q_{ijk} - q_{i,j,k-2}) + 2(2q_{i,j,k+1} - q_{i+1,j,k+1} - q_{i-1,j,k+1}) \\ &\quad + (2q_{i,j,k+1} - q_{ijk} - q_{i,j,k+2})]. \end{aligned} \quad (25)$$

Actually this is not as complex as it seems. With the lattice constant  $a$ , we can transform the above expression to be

$$A_{ijk}^{(1)} = \frac{3a^2}{b+K} \left( \frac{b}{6} - K \right) \nabla^2 q(\mathbf{r}, t). \quad (26)$$

The second part,  $A_{ijk}^{(2)}$ , comes from the redistribution between  $\sigma_{ijk}$  and the farther spins,  $\sigma_{i'j'k'}$ . The result is

$$\begin{aligned} A_{ijk}^{(2)} &= -(N-1)q_{ijk}(t) + \frac{1}{2} \sum_{i'j'k' \neq ijk} [q_{ijk}(t) + q_{i'j'k'}(t)] \\ &+ \frac{K}{2b} \sum_{i'j'k' \neq ijk} \sum_w [(q_{i+w,j,k}(t) + q_{i,j+w,k}(t) \\ &\quad + q_{i,j,k+w}(t)) - (q_{i'+w,j',k'}(t) + q_{i',j'+w,k'}(t) \\ &\quad + q_{i',j',k'+w}(t))], \end{aligned} \quad (27)$$

Substituting Eqs. (26) and (27) into Eq. (22), we obtain the evolving equation of the Gaussian model built on 3D adding-type SWN,

$$\frac{\partial q(\mathbf{r},t)}{\partial t} = \left[ \frac{3a^2}{b+K} \left( \frac{b}{6} - K \right) + p_A(N-1) \frac{a^2 K}{2b} \right] \nabla^2 q(\mathbf{r},t) + \frac{p_A(N-1)}{2} \left( 1 - \frac{6K}{b} \right) [M(t) - q(\mathbf{r},t)]. \quad (28)$$

Similarly, on 3D rewiring-type SWN we obtain

$$\frac{\partial q(\mathbf{r},t)}{\partial t} = \left\{ (1-p_R) \frac{3a^2}{b+K(1-p_R)} \left[ \frac{b}{6} - K(1-p_R) \right] + 3p_R \frac{K}{2b} (1-p_R) \right\} \nabla^2 q(\mathbf{r},t) + \frac{3p_R}{2} \left[ 1 - \frac{6K}{b} \right] \times (1-p_R) [M(t) - q(\mathbf{r},t)]. \quad (29)$$

From these two equations we can clearly see the influence of the system built on SWN as a whole on individual spins. On regular lattices the evolution of the system can be explained by the diffusion mechanism, while on SWN, to some degree, the individual spins will automatically adjust itself to approach the average magnetization. This is obviously the result of the global coherence introduced by the small fraction of the long-range bonds.

On regular lattice, where the evolution is pure diffusion,  $\partial q(\mathbf{r},t)/\partial t = \mathcal{D} \nabla^2 q(\mathbf{r},t)$ , and the diffusion coefficient  $\mathcal{D}$  will vanish near the critical point. However, on adding-type SWN, where  $\partial q(\mathbf{r},t)/\partial t = \mathcal{D}' \nabla^2 q(\mathbf{r},t) + \mathcal{C}[M(t) - q(\mathbf{r},t)]$ , two temperatures,  $T_{c1}^A$  and  $T_{c2}^A$  can be obtained by setting  $\mathcal{D}'$  and  $\mathcal{C}$  to zero, respectively.  $T_{c1}^A$  will be lower than the critical temperature on a regular lattice,  $T_c^{reg}$ , and suggests that the point at which the diffusion stops will be lowered by the randomness.  $T_{c2}^A$  equals  $T_c^{reg}$ , and at this point the evolution will be pure diffusion. Similarly, there are also two temperatures for rewiring-type SWN, but we will have  $T_{c1}^R < T_{c2}^R < T_c^{reg}$ .

Obviously, the system behavior strongly depends on the temperature. For example, we study a one-dimensional system, and the initial magnetization is  $q(x,0) = \sin x$ . When the temperature  $T > T_{c2}$ , both  $\mathcal{D}'$  and  $\mathcal{C}$  are positive, and the magnetization will approach homogeneity. When  $T < T_{c1}$ , both  $\mathcal{D}'$  and  $\mathcal{C}$  are negative, and the inhomogeneity gets more remarkable during the evolution. When  $T_{c2} < T < T_{c1}$ ,  $\mathcal{D}' > 0$  but  $\mathcal{C} < 0$ , and this is a more complex region. Although here we can still easily predict the system behavior with Eq. (28) and obtain a stationary point, generally the evolution will be strongly dependent on the local magnetization.

### C. Application to the Ising model

The second application of the simplified method is to the one-dimensional ferromagnetic Ising model governed by the Glauber-type mechanism. Recently, it has been studied both

analytically [4,9] and with Monte Carlo simulations [10–14]. Due to the mathematical difficulties, the study was not going very smoothly at the beginning. In Ref. [9], Gitterman concluded that the random long-range interactions, the number of which is above a minimal value, lead to a *phase transition*. In Ref. [4], Barrat and Weigt used some approximations and expected a finite critical point “at least for sufficiently large  $p$  and  $k \geq 2$ ” on rewiring-type SWN. They found out that *this transition is of the mean-field type*. Although they could not calculate the transition analytically, the numerical computation demonstrated a *nonvanishing order parameter in the presence of a vanishingly small fraction of shortcuts*, for  $k=2$  and 3. The numerical results seemed to support the following relationship,  $T_c \sim -2k/\log_{10} p_R$ . The more recent Monte Carlo studies have proved the above-mentioned conclusions on adding-type SWN and rewiring-type SWN, respectively, but in those cases  $k=1$ . The analysis of the relationship between  $T_c$  and  $p_R$  can be found in Ref. [11]. The critical exponents obtained from the simulation [11,12,15],  $\beta \approx 1/2$ ,  $\alpha \approx 0$ , and  $\nu \approx 1/2$ , further establish the mean-field character. However, although there is substantial numerical proof, the nonexistence of such a threshold  $p_c$ , for a finite-temperature transition, is still to be confirmed.

In this paper, we will apply the simplified method, which is of the mean-field nature, to study the dynamic properties of the one-dimensional Ising model built on both adding-type and rewiring-type SWN. We hope the success in the Gaussian model will continue in Ising system, of which the behavior is already known to be the mean-field type. Although we cannot obtain the full picture of the evolution, we are able to get the exact  $p$  dependence of the critical point, and some interesting critical exponents. As will be shown below, our result agrees perfectly with the above-mentioned numerical simulation.

We find that the system shows very similar behavior for adding- and rewiring-type SWN. We shall give the details of rewiring-type SWN only, and report on the results of adding-type SWN later.

For a rewiring-type SWN, the effective Hamiltonian is the same as Eq. (20) (the one-dimensional version). We substitute it into the single-spin evolving equation, Eq. (3), and obtain

$$\frac{dq_k(t)}{dt} = -q_k(t) + \sum_{\{\sigma\}} \tanh[K(1-p_R)(\sigma_{k-1} + \sigma_{k+1}) + 2Kp_R \bar{\sigma}] P(\{\sigma\}; t), \quad (30)$$

where  $\bar{\sigma} = \sum_{j \neq k} \sigma_j / (N-1) \approx \sum_j \sigma_j / N$ .

If  $p_R=0$ , then it is the one-dimensional Ising model on regular lattice, which we are familiar with. One can continue to write

$$\frac{dq_k(t)}{dt} = -q_k(t) + \frac{1}{2} (q_{k-1} + q_{k+1}) \tanh 2K$$

and

$$\frac{dM(t)}{dt} = -M(t)(1 - \tanh 2K).$$

It yields  $M(t) \propto e^{-t/\tau}$ , where  $\tau = (1 - \tanh 2K)^{-1}$ . When  $K \rightarrow K_c^{reg} = \infty$ ,  $\tau \sim \xi^z \sim (e^{2K})^2$ , and thus  $z = 2$ .

If the rewiring probability  $p_R = 1$ , then

$$\frac{dq_k(t)}{dt} = -q_k(t) + \sum_{\{\sigma\}} \tanh(2K\bar{\sigma})P(\{\sigma\};t),$$

and

$$\frac{dM(t)}{dt} = -M(t) + \sum_{\{\sigma\}} \tanh(2K\bar{\sigma})P(\{\sigma\};t).$$

Although  $\langle \tanh(2K\bar{\sigma}) \rangle \neq \tanh(2KM)$ , if we study the case when, near the critical point, the system is in almost thorough disorder, we will have  $\langle \tanh(2K\bar{\sigma}) \rangle \sim \tanh(2KM) \sim 2KM$ , and

$$\frac{dM(t)}{dt} \sim -M(t) + \tanh[2KM(t)].$$

This helps us to determine the critical point  $K_c = 1/2$ . When  $K < K_c = 1/2$ , then the system will be stable in a disordered state with  $M = 0$ , but when  $K \geq K_c = 1/2$ , there appears some kind of order. Taking Taylor expansion, one can find that when  $K$  is near  $K_c$ ,  $M \sim (K - K_c)^{1/2}$ . This leads to  $\beta = 1/2$  (in this specific situation). We also get the relaxation time  $\tau \sim |K - K_c|^{-1}$ , with the scaling hypothesis (see below) we have  $z\nu = 1$ .

Qualitatively similar situation should also be found when  $p_R$  is between 0 and 1. First we assume that there is a critical temperature, above which the system is disordered and below which there begins to show nonzero magnetization. From Eq. (30) we find that if initially  $M = 0$  is given, then the system will stay in this disordered state. But below the critical temperature this equilibrium will not be stable. We can determine the critical point introducing a small perturbation. When  $M \rightarrow 0$ ,

$$\begin{aligned} \frac{dq_k(t)}{dt} &\approx -q_k(t) + \sum_{\{\sigma\}} \tanh[K(1-p_R)(\sigma_{k-1} \\ &+ \sigma_{k+1})]P(\{\sigma\};t) + 2Kp_R \sum_{\{\sigma\}} \bar{\sigma} \{1 - \tanh^2[K(1 \\ &- p_R)(\sigma_{k-1} + \sigma_{k+1})]\}P(\{\sigma\};t) \\ &= -q_k(t) + \frac{1}{2}(q_{k-1} + q_{k+1})\tanh[2K(1-p_R)] \\ &+ 2Kp_R \sum_{\{\sigma\}} \bar{\sigma} \left\{ 1 - \frac{1}{2}(1 + \sigma_{k-1}\sigma_{k+1})\tanh^2[2K(1 \\ &- p_R)] \right\} P(\{\sigma\};t). \end{aligned} \quad (31)$$

Because  $M$  is very small, we believe  $1/N \sum_k \langle \bar{\sigma} \sigma_{k-1} \sigma_{k+1} \rangle$  is an even smaller quantity of higher order. Thus

$$\begin{aligned} \frac{dM(t)}{dt} &= -M + M \tanh[2K(1-p_R)] \\ &+ 2Kp_R \left\{ 1 - \frac{1}{2} \tanh^2[2K(1-p_R)] \right\} M. \end{aligned} \quad (32)$$

The critical point can be determined as

$$\begin{aligned} \tanh[2K_c^R(1-p_R)] + 2K_c^R p_R \left\{ 1 - \frac{1}{2} \tanh^2[2K_c^R(1-p_R)] \right\} \\ = 1. \end{aligned} \quad (33)$$

If  $K < K_c^R$ , the disordered state  $M = 0$  will be stable, but if  $K > K_c^R$ , a small perturbation will drive the system apart from the disordered state towards nonzero magnetization. Now we continue to find several interesting critical exponents.

(1)  $\chi \sim |T - T_c|^{-\gamma}$ : Near the critical point,  $M_{eq} \rightarrow 0$ . If a weak field  $H$  is introduced, we will have

$$-\beta\mathcal{H} = \sum_k \sigma_k \left[ K(1-p_R)\sigma_{k+1} + p_R \frac{K}{N-1} \sum_{j \neq k} \sigma_j + \frac{H}{k_B T} \right],$$

and

$$\begin{aligned} \frac{dq_k(t)}{dt} &= -q_k(t) + \sum_{\{\sigma\}} \tanh \left[ K(1-p_R)(\sigma_{k-1} + \sigma_{k+1}) \right. \\ &\left. + 2Kp_R \bar{\sigma} + \frac{H}{k_B T} \right] P(\{\sigma\};t). \end{aligned} \quad (34)$$

Following the same way as that taken in the calculation of the critical point (in fact, we can just replace  $2Kp_R \bar{\sigma}$  by  $2Kp_R \bar{\sigma} + H/k_B T$ ), we get

$$\begin{aligned} \frac{d}{dt} M(t) &= -M(t) + M(t) \tanh[2K(1-p_R)] + \left( 2Kp_R M(t) \right. \\ &\left. + \frac{H}{k_B T} \right) \left\{ 1 - \frac{1}{2} \tanh^2[2K(1-p_R)] \right\}. \end{aligned} \quad (35)$$

From Eq. (35), one can easily find that in a system in equilibrium near the critical point,  $K = K_c^R + \Delta$ , and

$$\chi \equiv \frac{\partial M}{\partial H} \sim \Delta^{-1}. \quad (36)$$

Thus,  $\gamma = 1$ .

(2)  $M \sim |T - T_c|^\beta$ : When  $M \rightarrow 0$ , we take

$$\frac{1}{N} \sum_k \langle \bar{\sigma} \sigma_{k-1} \sigma_{k+1} \rangle \sim M^3.$$

Then from Eq. (31) we can get



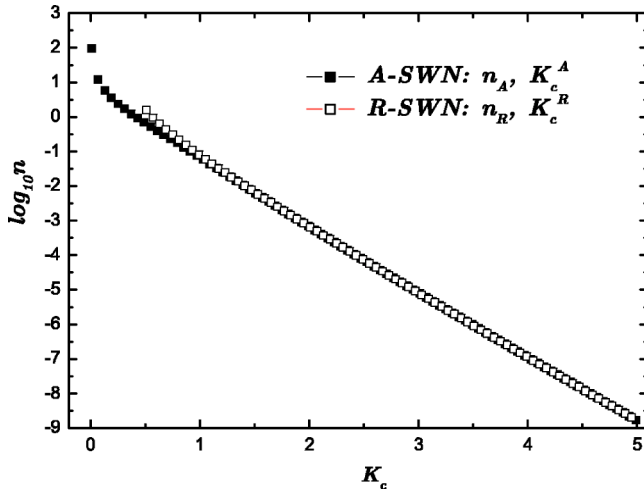


FIG. 1. The  $n$  dependence of critical point  $K_c$  of 1D Ising model on adding-type and rewiring-type SWN.

$$\begin{aligned} \frac{dM(t)}{dt} \approx & - \left( 1 - \tanh[2K(1-p_R)] - 2Kp_R \right. \\ & \times \left. \left\{ 1 - \frac{1}{2} \tanh^2[2K(1-p_R)] \right\} \right) M(t) \\ & - Kp_R \tanh^2[2K(1-p_R)] M^3(t). \end{aligned}$$

When  $K - K_c^R = \Delta \rightarrow 0$ , let  $dM(t)/dt = 0$ , and one will get  $M^2 \sim \Delta$  by taking Taylor expansion. Thus, the critical exponent  $\beta = 1/2$ .

(3)  $\tau \sim \xi^z \sim |T - T_c|^{-z\nu}$ : When studying the critical slowing down, we can assume  $M$  to be very small, and thus we use Eq. (32). It yields

$$M(t) = M(0)e^{-t/\tau},$$

where

$$\begin{aligned} \tau^{-1} = & 1 - \tanh[2K(1-p_R)] - 2Kp_R \\ & \times \left\{ 1 - \frac{1}{2} \tanh^2[2K(1-p_R)] \right\}. \end{aligned}$$

If  $K \rightarrow K_c^R$ , then  $\tau \rightarrow \infty$ . If  $K = K_c^R - \Delta$ , and  $\Delta \rightarrow 0$ , then one will find  $\tau \sim \Delta^{-1}$ . Thus  $z\nu = 1$ . It has been found in Monte Carlo simulations that  $\nu \approx 1/2$ , so  $z = 2$ . It is of the same value as that obtained on the regular lattice.

1D Ising model on adding-type SWN show *qualitatively the same* behavior Its critical point can be determined as

$$\tanh 2K_c^A + K_c^A(N-1)p_A \left( 1 - \frac{1}{2} \tanh^2 2K_c^A \right) = 1, \quad (37)$$

and we have found *the same critical exponents*,  $\gamma$ ,  $\beta$ , and  $z$ .

For a vertex on an adding-type SWN lattice, the number of the long-range bonds is  $n_A \sim (N-1)p_A$ , while for a vertex on a rewiring-type SWN lattice,  $n_R \sim 2p_R$ . The  $n$  dependence of critical point  $K_c$  of 1D Ising model on adding-type and rewiring-type SWN, Eqs. (33) and (37), can be found in

Fig. 1. One can clearly see the above mentioned approximate relationship,  $K_c^A \propto -\log_{10} n_A \sim -\log_{10}[(N-1)p_A]$ , or  $K_c^R \propto -\log_{10} n_R \sim -\log_{10}(2p_R)$ .

When  $n_A$  and  $n_R$  are small enough (in the small-world region), from Eqs. (33) and (37) we can get

$$\begin{aligned} n_R \approx & (1 - \tanh 2K_c^R) / \left( \frac{1}{2} K_c^R \tanh^2 2K_c^R \right), \\ n_A \approx & (1 - \tanh 2K_c^A) / \left[ K_c^A \left( 1 - \frac{1}{2} \tanh^2 2K_c^A \right) \right]. \end{aligned}$$

When  $n_A$  and  $n_R$  are approaching zero,  $K_c^{A,R} \rightarrow \infty$ . For the same value of the critical point,  $(n_R - n_A) \rightarrow 0^+$ . As shown in Fig. 1, for most of the region, the two curves are very close to each other. However, though the difference may be infinitesimal, the curve of  $n_R$  is always above that of  $n_A$ , as is distinct when these curves are relatively large.

## VI. THE INFLUENCE OF THE RANDOMNESS ON THE CRITICAL POINT

Regarding the behavior of the critical point, our results show interesting contrast between the Gaussian model and the Ising model. Here, we shall mention another interesting model system, the  $D$ -dimensional mean-field (MF) Ising model, which might help us understand this problem. For a randomly selected spin in this model, each of its  $2D$  nearest neighbors is replaced by an averaged one, and it is well known that the critical point is  $K_c = 1/2D$ . A simple calculation will yield that on rewiring-type SWN,  $K_c^R$  will not change, while on adding-type SWN, since the long-range bonds increase the contact of a spin with the system,  $K_c^A = 1/(2D + n_A)$ .

Our result of the 1D Ising model is in contrast to that of the MF Ising model and the Gaussian model, while for each one of them the critical temperature will be very close on adding-type SWN and rewiring-type SWN (the former will be higher) in the small-world region. This may be a result of the totally different role played by the long-range bonds. (1a) In the  $D$ -dimensional MF Ising model, the critical point is solely determined by the mean coordination number that decides the coupling between an individual spin and the system. It is unchanged on rewiring-type SWN but will increase on adding-type SWN. Thus, the critical temperature will stay unaltered on rewiring-type SWN but will be slightly increased by the long-range bonds on adding-type SWN, which typically take up only a small fraction. (1b) As is shown by earlier studies [19], the Gaussian model, though being a very different system, has a critical point that also *only depends on the mean coordination number*. On SWN, its critical point is very similar to the MF Ising model. It is believed to be mainly a mathematical result, and to understand this we shall review the calculations in Sec. IV. On rewiring-type SWN, for an individual spin, the long range interaction partly replaces the nearest-neighbor (nn) coupling. However, the coordination number can be considered unchanged, since no bonds are created or eliminated (they are just redirected). As a result, on the right-hand side of the

evolving equations of the spins, the lost part of the nn coupling is exactly compensated by the MF term. The critical point is determined by taking average of the evolving equations of the spins, which only consist of linear terms. Thus, it is mathematically straightforward that the critical point will stay the same. On adding-type SWN, the consideration is similar, except that here the coordination number will be slightly increased. (2) The crossover observed in the one-dimensional Ising model is certainly governed by a different mechanism. For example, on rewiring-type SWN, there are two competing length scales: the correlation length  $\xi \sim \exp(2J/k_B T)$ , and the characteristic length scale of the SWN, which can be taken as the typical distance between the ends of a shortcut [25],  $\zeta \sim p_R^{-1/D} = p_R^{-1}$ . When  $\xi < \zeta$ , the system basically behaves as a regular lattice, otherwise it will show MF behavior as an effect of the long range interactions. The transition occurs at  $\xi \approx \zeta$ , suggesting a critical point  $T_c \sim |\log_{10} p_R|^{-1}$ . Since the two structures, rewiring-type SWN and adding-type SWN, have the same length scales, the critical point can hardly be separated in the small-world region. However, interestingly we find that the mechanism in Eq. (1), i.e., long-range bonds  $\rightarrow$  coordination number (interaction energy)  $\rightarrow$  critical point, can also be observed, though being a minor factor. As is mentioned in Sec. V C, for the same expected number of long-range bonds, the critical temperature on adding-type SWN will be higher than that obtained on rewiring-type SWN. We expect it to be a general phenomenon, though not yet reported by the numerical simulations.

## VII. SUMMARY AND DISCUSSION

In this paper, we study the critical dynamics, Glauber type and Kawasaki type, on two typical small-world networks, adding type and rewiring type.

*The logical sequence.* As the introductory content we discuss the general approach of the critical dynamics, which is theoretically straightforward but may be mathematically too complex. We directly apply it to the kinetic Gaussian model governed by Glauber-type mechanism and obtain its evolution. We observe that, in the *dynamic* evolution of the individual spins, the influence of the system as a whole, which is the result of the presence of the long-range bonds, takes the mean-field-like form as if it comes from an averaged spin. At the same time, earlier studies have revealed the (MF) *static* behavior of the Ising model and X-Y model. These both suggest us to present the following simplified method. All the SWN realizations are deemed as a single one, with both the effective Hamiltonian and the effective behavior averaged over all of them. It is tested in the same model and exactly leads to the same rigorous result. Then this method, which is believed to be theoretically reliable and mathematically feasible, is applied to two more difficult problems, the Gaussian model governed by Kawasaki-type mechanism and the one-dimensional kinetic Ising model.

*The Gaussian model.* (1) Whether it is built on adding-type SWN or rewiring-type SWN, the long-range bonds introduce the mean-field-like global influence of the system to the dynamic evolution of the individual spins. On adding-

type SWN such influence is additional while on rewiring-type SWN, it partly replaces that of the neighboring spins. (2) On adding-type SWN, as more long-range bonds are added, the critical temperature gets higher; but on rewiring-type SWN it does not differ at all from that of the regular lattice. This interesting discrepancy is explained in Sec. VI. (3) In both networks, the relaxation time  $\tau = 1/(1 - K/K_c)$ , and thus  $z\nu = 1$ . This dynamic property has also been obtained on regular lattices and fractal lattices [19–22]. It is highly universal, independent of the geometric structure and the dynamic mechanism. (4) On SWN, the evolution of the Kawasaki-type model can be viewed as the combination of two mechanisms, the diffusion and the automatic adjustment of the single spin to approach the average magnetization. The pure diffusion equation  $(\partial/\partial t)q(\mathbf{r}, t) = D\nabla^2 q(\mathbf{r}, t)$  will be modified as  $(\partial/\partial t)q(\mathbf{r}, t) = D'\nabla^2 q(\mathbf{r}, t) + C[M(t) - q(\mathbf{r}, t)]$ . By setting  $D'$  and  $C$  to zero we will get two competing characteristic temperatures, instead of the single definition of the critical point for the regular lattices ( $D=0$ ). The temperature dependence of the dynamic evolution is discussed in Sec. V B.

*The Ising model* analytically studied by the simplified method: The system shows very similar behavior on the two networks, adding-type SWN and rewiring-type SWN. Introducing a very small perturbation of the local magnetization to the disordered state, we obtain the critical point by judging the stability of the equilibrium. The inexistence of such a threshold  $p_c$  for a finite-temperature transition is confirmed.<sup>1</sup> From the dynamic equation, we obtain the critical exponents  $\gamma$  and  $\beta$  in agreement with the numerical simulation and the already-proved MF behavior of the system. The relaxation time is divergent near the critical point as  $\tau \sim |T - T_c|^{-1}$ , and thus  $z\nu = 1$  (note the same relationship in the Gaussian model). In the 1D Ising model, the SWN effect does not change the dynamic critical exponent  $z = 2$ .

*The influence of the randomness on the critical point.* Our result of the 1D Ising model is in contrast to that of the MF Ising model and the Gaussian model. For each one of them the critical temperature on adding-type SWN,  $T_c^A$ , will be higher than that on rewiring-type SWN,  $T_c^R$ , obtained for the same expected number of the long-range bonds, though the difference may be hardly perceivable in the small-world region. A detailed analysis of the responsible mechanisms can be found in Sec. VI.

*Prospects.* We hope that further studies on critical dynamics will continue to reveal interesting dynamic characteristics of the widely existing critical phenomena combined with SWN. The simplified method shall become a useful tool in this field. As the phase transition is of the MF nature, this method is certainly the best choice, and in this sense it is basically different from the custom MF approximation applied to other universality classes. Presently, besides the numerical study, analytical treatment in the dynamic aspect is

<sup>1</sup>As mentioned above, for some physical considerations one may define a different  $J'$  for the long-range bonds, which might be much smaller than  $J$ . With this method we can prove that there is not such a threshold  $J'_c$  either.

scarce compared to the study of the static properties [4,9]. In fact, at least in some cases, such a study may be feasible and useful indeed. Our work, especially on the Ising model, also shows that the study of the dynamic aspect is often able to yield much information on the general properties in a relatively convenient way.

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#### APPENDIX A: 3D GAUSSIAN MODEL OF THE REWIRING-TYPE SWN

(1). The two regular bonds on  $\sigma_k$  (connecting  $\sigma_{k-1}$  and  $\sigma_{k+1}$ ) have not been rewired, and the probability is  $(1-p)^2$ . This group can be further divided into many subgroups:

(1a) There are no random bonds on  $\sigma_k$ . The probability is  $(1-p)^2(1-[1/(N-1)p])^{N-2}$ ,

$$\frac{dq_k}{dt} = -q_k + \frac{K}{b}(q_{k-1} + q_{k+1}).$$

(1b) There is only one random bond on  $\sigma_k$ . The probability is

$$(1-p)^2 C_{N-2}^1 \frac{1}{N-1} p \left(1 - \frac{1}{N-1} p\right)^{N-3},$$

and

$$\frac{dq_k}{dt} = -q_k + \frac{K}{b}(q_{k-1} + q_{k+1} + M).$$

(1c) There are  $n$  random bonds on  $\sigma_k$ , and the probability is

$$(1-p)^2 C_{N-2}^n \left(\frac{1}{N-1} p\right)^n \left(1 - \frac{1}{N-1} p\right)^{N-n-2}.$$

This group can be further divided into  $C_{N-1}^n$  subgroups, each corresponding to  $n$  specific spins connected to  $\sigma_k$  via the random bonds, and with the same probability

$$(1-p)^2 \left(\frac{1}{N-1} p\right)^n \left(1 - \frac{1}{N-1} p\right)^{N-n-2}.$$

In the  $i$ th subgroup,

$$\frac{dq_k}{dt} = -q_k + \frac{K}{b} \left( q_{k-1} + q_{k+1} + \sum_{i=1}^n q_{i_l} \right).$$

Thus, averaging them we get

$$\frac{dq_k}{dt} = -q_k + \frac{K}{b}(q_{k-1} + q_{k+1} + nM).$$

(1d) There are  $N-2$  random bonds on  $\sigma_k$ , and the probability is  $(1-p)^2[1/(N-1)p]^{N-2}$ ,

$$\begin{aligned} \frac{dq_k}{dt} &= -q_k + \frac{K}{b} \left( q_{k-1} + q_{k+1} + \sum_{i=1}^{N-2} q_{i_l} \right) \\ &= -q_k + \frac{K}{b} [q_{k-1} + q_{k+1} + (N-2)M]. \end{aligned}$$

Thus, the first part of the time derivative of single spin, is

$$\begin{aligned} \left( \frac{dq_k}{dt} \right)_1 &= (1-p)^2 \left[ -q_k + \frac{K}{b}(q_{k-1} + q_{k+1}) \right. \\ &\quad + \frac{K}{b} \sum_{n=0}^{N-2} C_{N-2}^n \left( \frac{p}{N-1} \right)^n \\ &\quad \left. \times \left( 1 - \frac{1}{N-1} p \right)^{N-n-2} nM \right]. \end{aligned} \quad (\text{A1})$$

(2) The bond connecting  $\sigma_{k-1}$  and  $\sigma_k$  is left unchanged, but that connecting  $\sigma_k$  and  $\sigma_{k+1}$  has been rewired. This bond can be rewired to each one of the  $N-1$  spins with equal probability  $(1-p)p/N$ . In each case, we can analyze the situation in the way described above. For example, the bond is rewired to  $\sigma_j$ .

(2a) There are no random bonds on  $\sigma_k$ . The probability is

$$(1-p) \frac{p}{N} (1-p)^2 \left( 1 - \frac{1}{N-1} p \right)^{N-2}.$$

Also,

$$\frac{dq_k}{dt} = -q_k + \frac{K}{b}(q_{k-1} + q_j).$$

(2b) There is only one random bond on  $\sigma_k$ . The probability is

$$(1-p) \frac{p}{N} C_{N-2}^1 \frac{1}{N-1} p \left( 1 - \frac{1}{N-1} p \right)^{N-3}.$$

Also,

$$\frac{dq_k}{dt} = -q_k + \frac{K}{b}(q_{k-1} + q_j + M).$$

(2c) There are  $n$  random bonds on  $\sigma_k$ , and the probability is

$$(1-p) \frac{p}{N} C_{N-2}^n \left( \frac{1}{N-1} p \right)^n \left( 1 - \frac{1}{N-1} p \right)^{N-n-2}.$$

Also

$$\frac{dq_k}{dt} = -q_k + \frac{K}{b}(q_{k-1} + q_j + nM).$$

(2d) There are  $N-2$  random bonds on  $\sigma_k$ , and the probability is  $[1/(N-1)p]^{N-2}$ . Also

$$\frac{d}{dt}q_k = -q_k + \frac{K}{b}(q_{k-1} + q_j + (N-2)M).$$

Thus the second part of the time derivative of single spin is

$$\begin{aligned} \left(\frac{dq_k}{dt}\right)_2 &= (1-p)p \left[ -q_k + \frac{K}{b}(q_{k-1} + M) \right. \\ &\quad \left. + \frac{K}{b} \sum_{n=0}^{N-2} C_{N-2}^n \left(\frac{p}{N-1}\right)^n \right. \\ &\quad \left. \times \left(1 - \frac{1}{N-1}p\right)^{N-n-2} nM \right]. \end{aligned} \quad (\text{A2})$$

(3) The bond connecting  $\sigma_k$  and  $\sigma_{k+1}$  is left unchanged, but that connecting  $\sigma_{k-1}$  and  $\sigma_k$  has been rewired (we omit the very small probability that this bond may be “rewired” to  $\sigma_k$ ). Now there is only one regular bond on  $\sigma_k$ . [Pay attention that this case is different from Eq. (2).] Based on a similar consideration, we have the third part of the time derivative of single spin which obeys

$$\begin{aligned} \left(\frac{dq_k}{dt}\right)_3 &= (1-p)p \left[ -q_k + \frac{K}{b}q_{k+1} + \frac{K}{b} \sum_{n=0}^{N-2} C_{N-2}^n \left(\frac{p}{N-1}\right)^n \right. \\ &\quad \left. \times \left(1 - \frac{1}{N-1}p\right)^{N-n-2} nM \right]. \end{aligned} \quad (\text{A3})$$

(4) Both the bonds connecting  $\sigma_k\sigma_{k+1}$  and  $\sigma_{k-1}\sigma_k$  have been rewired. Based on similar consideration, we have the fourth part of the time derivative of single spin,

$$\begin{aligned} \left(\frac{dq_k}{dt}\right)_4 &= p^2 \left[ -q_k + \frac{K}{b}M + \frac{K}{b} \sum_{n=0}^{N-1} C_{N-2}^n \left(\frac{p}{N-1}\right)^n \right. \\ &\quad \left. \times \left(1 - \frac{1}{N-1}p\right)^{N-n-2} nM \right]. \end{aligned} \quad (\text{A4})$$

Applying Eq. (10), we get

$$\begin{aligned} \frac{dq_k}{dt} &= \sum_{i=1}^4 \left(\frac{dq_k}{dt}\right)_i \\ &= -q_k + p \frac{2K}{b}M + (1-p) \frac{K}{b}(q_{k-1} + q_{k+1}). \end{aligned} \quad (\text{A5})$$

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